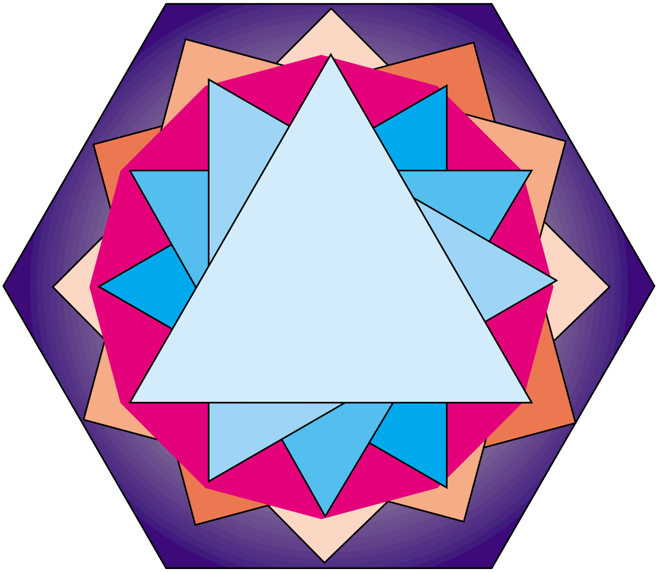
Maths Workbook 2

Number Patterns : 2



You should be able to work through this text by yourself. If you get stuck on an example, ASK ...; but do not ask until you have *thought* about the example.

DO NOT WRITE IN THIS BOOKLET

Copy the exercises into your Maths notebook

Introduction to "Number Patterns"

This is the second in the series of workbooks written for pupils who needed extra material in the mixed ability class setting.

It completes the work on the Arithmetic Progression started in Workbook 1, leading the pupil on to discover new patterns that can be formed from the Arithmetic Progression.

This second book was issued in May 1978 for pupils in the First Forms at Trinity School, Carlisle, but was also used at times with other years when additional work was called for.

This (slightly revised) edition is offered to anyone who will find it of use. All comments welcome!

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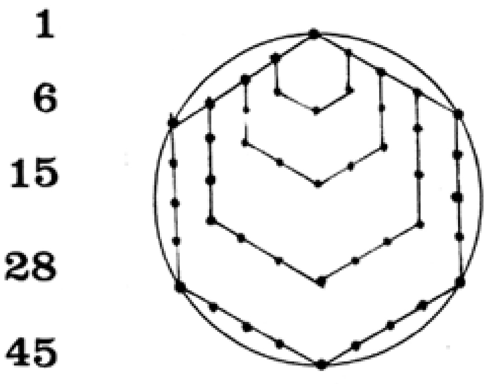


Number patterns 2

There is a connection between the square numbers and the APs we have looked at.

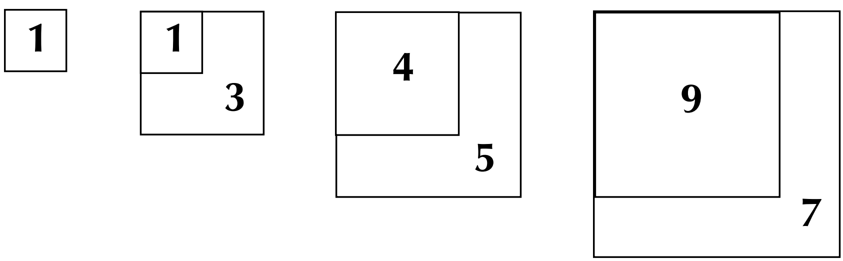
The set of square numbers {1, 4, 9, 16 ,...} does not form an AP as the differences between the successive terms are not the same.

4 - 1 = 3; 9 - 4 =5; 16 - 9 = 7; 25 - 16 = 9; 36 - 25 = 11, and the differences, in order, are 3, 5, 7, 9, 11.

These differences are part of the set of odd numbers, which is an AP. And this in turn tells us that we can form the set of square numbers by adding the odd numbers:

1	= 1 x 1	= 1^2
1 + 3 = 4	= 2 x 2	= 2^2
1 + 3 +5 = 9	= 3 x 3	= 3^2
1 + 3 + 5 + 7 = 16	= 4 x 4	= 4^2
1 + 3 + 5 + 7 + 9 = 25	= 5 x 5	= 5^2

and so on. (The first line for 1 has been added to complete the pattern).This is a pattern which we can also show as a diagram:



1   1+3=4   4+5=9   9+7=16

1^2 2^2 3^2 4^2

Each of the square numbers is obtained by adding the next odd number to the preceding square number.

From the set of odd numbers we have produced the set of square numbers, and we can illustrate this on a diagram as a series of squares. Recognising patterns in numbers is important; at the simplest level it makes our basic arithmetic easier. Do other progressions produce shapes we can draw in the same way?

We shall see what happens when we add the terms of other APs.

What else can we discover? Look again at the table connecting the odd numbers with the square numbers:

We add	that is	and we obtain
1	the first odd number	$\mathbf{1}^2$
1+3	the first **2** odd numbers	$\mathbf{2}^2$
1+3+5	the first **3** odd numbers	$\mathbf{3}^2$
1+3+5+7	the first **4** odd numbers	$\mathbf{4}^2$

Now ask yourself how many odd numbers in order did we add to get 3^2? and 4^2? so how many should we add to get 6^2?

Do we actually need to add any numbers to get the answer? Looking at the pattern we have found we can see that the square number we get depends on how many of the terms we add. If we add the first twenty odd numbers in order then we'll get the square of twenty.

If we add the first **n** odd numbers in order we get the square of **n**.The sum of the first **n** odd numbers is $\mathbf{n^2}$.

This is the sort of "spotting of patterns" that is important.

Let's look at another AP and the sums of its terms.

{1, 9, 17, 25, ...}

The first sum (and the first term) is 1

the second sum is 1 + 9 = 10

the third sum can be found by adding the next term in order to this last sum

17 + 10 = 27

and so on.

The first sum is 1, the second 10, the third 27, and these form a new set {1, 10, 27, 52, ...}.

Exercise 1

What is the sum of the first three terms of each of these APs?

1) 1, 2, 3, 4, 5, ...
2) 1, 4, 7, 10, 13 ...
3) 5, 11, 17, 23, ...

Here is another AP {1, 5, 9, 13 ...}

Term number	is	sum number	is
1	1	1	1
2	5	2	6
3	9	3	15
4	13	4	28

Make tables like this and show the first 4 sums of each of these:

4) 1, 10, 19, 28, ...
5) 1, 1½, 2, 2½, ...
6) 40, 38, 36, 34, ...

Secondary Words!

Now don't say "times", say "multiply",
And say "divide" for "share";
"Whole numbers" now are "integers" -
And less I couldn't care ...

In workbook 1 we labelled the terms of the AP with the letter **t**: t_1, t_2, t_n and so on. For the new series produced by adding the terms we can do the same, but since the terms produced are sums we will label them with the letter **s**.

The terms of the new series will be s_1, s_2 ... s_n

The table on the previous page can now be written as

term of AP	is	sum is	term of new (sum) set
t_1	1	1	s_1
t_2	5	6	s_2
t_3	9	15	s_3

and so on.

Exercise 2

For each set of numbers find the sum asked for

1) 1, 3, 5, ... find s_5
2) 1, 4, 7, ... find s_6
3) 1, 8, 15, ... find s_7
4) 100, 98, ... find s_5
5) 8, 6, 4, ... find s_3
6) 8, 4, 0,... find s_5
7) 1, 1⅓, 1⅔, ... find s_4
8) 2·5, 2, ... find s_8

After question 8 you will appreciate that where there are only a few terms to be added the amount of work involved in finding a given sum is not great; but if you want to add a lot of terms - say you wanted to find the 100th sum of the set 1, 81, 161, ...?

There must be a quicker way!

(Before you object that it wouldn't take long on a calculator remember (a) we're not letting you use a calculator and (b) there is still a much quicker way to do these whether you use a calculator or a bit of mental arithmetic).

Just as we discovered a quick way to find a given term of an AP (we called the general term t_n) so we can also discover a quick way to find a given sum (s_n) for the sum of n terms of an AP.

Back to square one, in a sense; let's look at the simplest set of numbers - the counting numbers - the AP {1, 2, 3, 4, ...}.

THE COUNTING NUMBERS AND THEIR SUMS

term	value	sum	value	=
t_1	1	s_1	1	1
t_2	2	s_2	1+2	3
t_3	3	s_3	1+2+3	6

continue this table up to t_{10} and s_{10}.

What set of numbers do the sums belong to?

Here is a different relationship to discover. Taking the terms of the sums (s_1, s_2, s_3 etc) from the example above add them in consecutive pairs,

e.g. $s_1 + s_2 = 1 + 3 = 4$, $s_2 + s_3 = 3 + 6 = 9$. Continue this idea as far as $s_9 + s_{10}$. To what set of numbers do these answers belong?

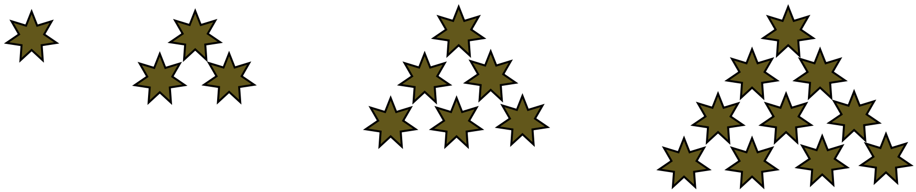
Now copy and complete this table

The sum of	is	which is
s_1 and s_2	4	2^2
s_2 and s_3	9	3^2
s_3 and s_4		
s_4 and s_5		

Now look at the subscripts (the small numbers attached to the s); what connection can you see between the subscripts and the number which is squared?

BACK TO THE COUNTING NUMBERS.

To which set do the sums of the counting numbers belong? You should have realised it was the set of “triangle numbers”. They take their name from the fact they can be drawn as triangles:

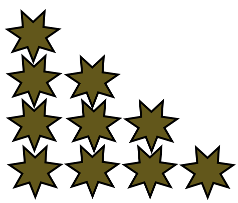


1 **3** **6** **10**

> And what mean all these mysteries to me
> Whose life is full of indices and surds?
> x2 + 7x + 53
> = 11/3.
>
> *(Lewis Carroll)*

This set of numbers, the triangle numbers, might have given you a clue to a way in which we can use their shape to find their values.

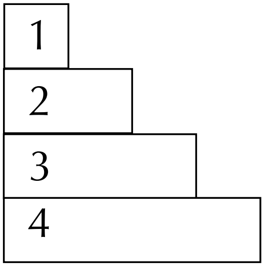
Here’s the fourth triangle number again, but turned into a more obvious position:



can we use this shape to find the sum? In the case of the square numbers we could see how the value of the square number corresponded to the area of the square shape.

Can we find the value of the triangle numbers by the same method - by finding the area of the triangle?

Here's the same triangle number drawn as a series of rectangles:



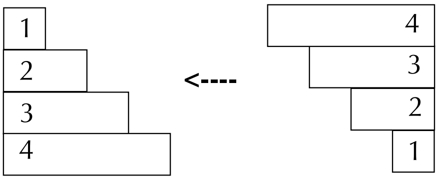
1

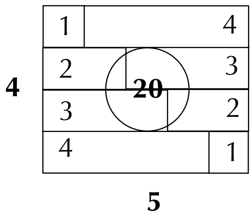
1+2=3

1+2+3=6

1+2+3+4=10

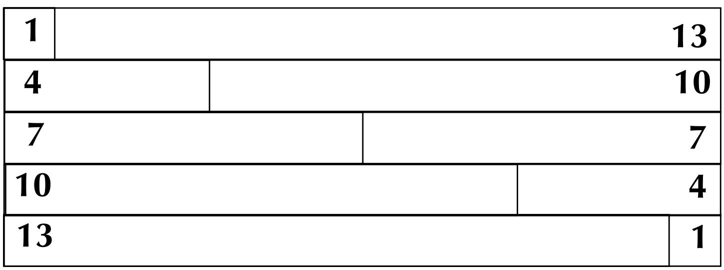
You should know how to calculate the area of a triangle. But let's go back to basics. You could think of the "triangle" above as being half of a rectangle; the area of a rectangle is found by multiplying the base by the height. Turn the triangle into a rectangle by sticking a copy on the end:





The rectangle has area 4 x 5 = 20; the triangle is half this, so the area is 10.

Let;s try the same idea with a different AP, {1, 4, 7...} and ask for s_5 by this strip and rectangle method:



Each strip is 14 units long. There are 5 strips, one for each of the first five terms in the AP. The area of the rectangle is 5 x 14 = 70, and so the area of the "triangle" is half of this, or 35; so $s_5 = 35$.

The diagram gives us other clues as to how we can simplify the work. The first strip is of length 14 and made up of the sum of 1 and 13. But 1 is the term t_1 and 13 is the last term (t_5) of the set written down to get s_5. There are 5 strips.

So all we need to know is the sum of the first and last terms (t_1 and t_5 here) and how many strips (5 because we want s_5) and then multiply the sum by the number of strips. As this is the rectangle all we do now is halve this answer for the triangle.

So for s_5 we have $(t_1+t_5) \times 5 = (1+13) \times 5 = 70$

and half this gives us $s_5 = 35$.

Check this with an example - we found the sum of the first 20 terms of the set of odd numbers to be 20^2, or 400.

We need to know the first term and the last term - in this case, t_1 and t_{20}. $t_1 = 1$. From our formula, $t_{20} = t_1 + 19d = 1 + 19(2) = 39$.

$t_1 + t_{20} = 1 + 39 = 40$

multiply by the number of "strips", 20

$20 \times 40 = 800$

and halve the answer

$800 \div 2 = 400$.

Exercise 3

Draw a strip diagram to find s_6 for each of these APs:

1) 4, 5, 6, ...
2) 1, 6, 11, ...
3) 12, 10, 8, ...

Exercise 4.

What is the sum of the first and last terms in these sets?

1) {1, 2, 3, 4, 5}
2) {2, 4, 6, 8, 10, 12}
3) {6, 4, 2, 0, -2}
4) {0·3, 0·6, 0·9, 1·2}
5) {¼, ½, ¾, 1, 1¼, 1½}

Another example.

In the AP {1, 10, 19, ...} what is the sum of the first 101 terms?
What is the first term? 1
What is the common difference? 9
What is the last term (t_{101})? $t_{101} = t_1 + 100(9) = 1 + 900 = 901$
Add the first term to the last: 1 + 901 = 902
multiply by the number of terms (101): 902 x 101 = 91,102
and lastly halve the answer: 91,102 ÷ 2 = 45,551.

Slightly quicker than adding up all the terms! It has often been said that mathematicians are lazy - they look for the easy way of doing everything.

For example, when we asked for the 20th odd number you could have used the formula, or you might have noticed another pattern. Since 39 is 1 less than 40 and 40 is twice 20 we could suggest that to get the 20th number we double 20 and knock one unit off, i.e. 20 x 2 = 40, 40 - 1 = 39. But is this true in general?

What's the 50th odd number? $t_{50} = t_1 + 49d = 1 + 98 = 99$.

Try the other way - double 50 and knock one off: 50 x 2 - 1 =99.

Check that this also works for some other odd numbers, say the third and the eighth.

And we find that in general the **n**th odd number is 2**n** - 1. This is an example of the way in which mathematicians think; they like to find recipes for making work easy. We found such a recipe for t_n, and now we'll find one for s_n.

Exercise 5

Use any method you like to find the sum asked for:

1) 1, 4, 7, ... find s_8
2) 1, 5, 9, ... fins s_{11}
3) 1, 6, 11, ... find s_{21}
4) 0·1, 0·2, 0·3, ... find s_9

To find a general formula look at what we did to find the sum

We added the first and last terms: $t_1 + t_n$

and multiplied this by the number of terms (n) $n(t_1+t_n)$

and then we halved the answer $(\frac{1}{2})n(t_1+t_n)$

So we have our formula for s_n:

$$\mathbf{s_n = \frac{n}{2}(t_1 + t_n)}$$

This also says that the sum of n terms is the number of terms, n, times the mean of the first and last term. (There is a link here, by the way, with the area of a trapezium)

Now if we tell you that for a certain AP the first term is 8 and the 10th is 0·2 and that we wanted the tenth sum, then you could find the mean of 8 and 0·2, which is 4·1, and multiply it by the number of terms, 10, and get $s_{10} = 10 \times 4{\cdot}1 = 41$.

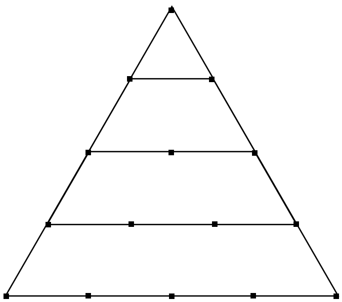
Exercise 5 continued

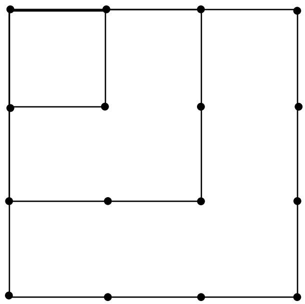
5) 100, 97, 94, ... find s_{11}
6) 8, 4, 0, ... find s_{12}
7) 1, 81, 161, 241, ... find s_{100}
8) 15, 14⅔, 14⅓, ... find s_{13}
9) 1, 8, 15, ... find s_{101}
10) 1, 5, 9, ... find s_{21}
11) 10, 11, 12, ... find s_{201}
12) 8, 10, 12, ... find s_{500}

There is another version of the formula for finding the sums of the terms of the AP, which we shall look at later. In traditional Algebra books it is usual to call the first term a and the last term l, and the formula is then $s_n = n/2(a+l)$.

THE SUMS AS GEOMETRICAL FIGURES.

The simplest set of numbers that we have is the set of counting numbers and we found that adding these gave us the triangle numbers. We also looked at the odd numbers and found their sums gave us the square numbers. Here are diagrams showing the sums as geometrical figures.





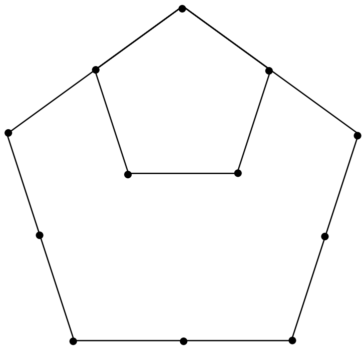
This seems to imply that we can show the sums of other progressions as polygons.

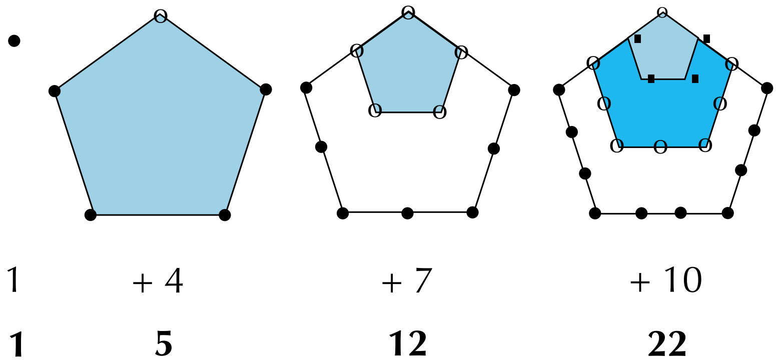
Triangle numbers come from the series {1, 2, 3, 4...} with d=1 and square numbers from {1, 3, 5, 7,...} with d=2. What about the series starting with 1 and going up in threes? {1, 4, 7, 10, ...}.

Series	Sums	d=	Type	Sides
1, 2, 3,	1, 3, 6,	1	triangle	3
1, 3, 5,	1, 4, 9.	2	square	4
1, 4, 7,	1, 5, 12	3	?	?

If there is a pattern then the next entry in “Type”, following the set of polygon names, should be “pentagon” and in “Sides” should be “5”.

Here is the pentagon set.
The diagram shows the first three *pentagonal* numbers 1, 5 and 12.





To go from the first pentagonal number (1) to the second (5) we add 4 points; from the second to the third we add 7; from the third to the fourth we add 10 - and so on, adding more terms from the AP {1, 4, 7, 10, ...}

And there is a connection between the number of sides of the polygon we can draw and the value of d:

d	figure	sides
1	triangle	3
2	square	4
3	pentagon	5

in each case the number of sides is two more than the value of d.

The sum of the terms of an AP that starts with 1 gives us a polygon - so these are called POLYGONAL NUMBERS.

> He thought he saw a triangle
> That squared its sides in glee,
> He looked again, and found it was
> A probability;
> It's all these decimals, he sighed,
> That spoil a body's tea.

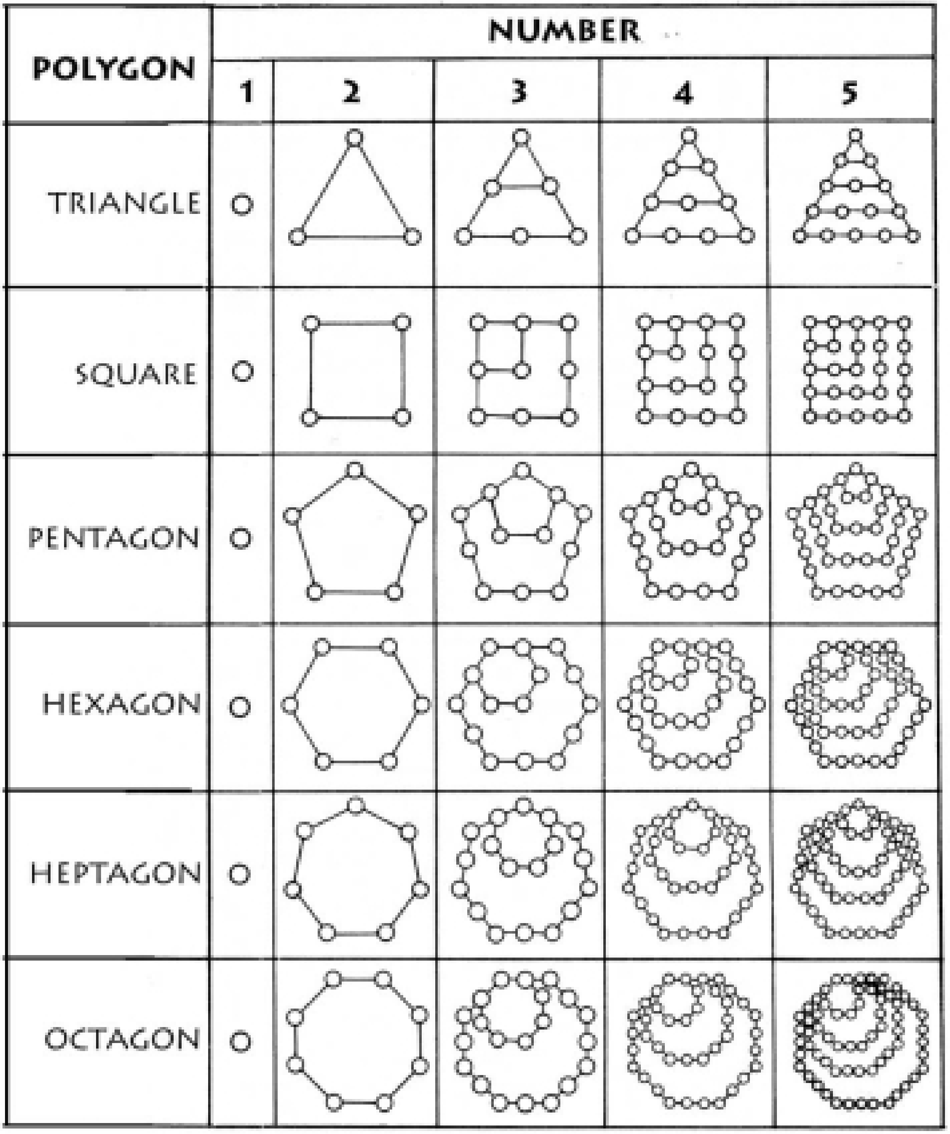
Exercise 6

1) Show the first three HEXAGONAL numbers as polygons
2) do the same for the first three OCTAGONAL numbers

THE OTHER VERSION OF THE SUMS FORMULA.
This is an alternative form that you might find useful.
We know that
$s_n = (n/2)(t_1 + t_n)$
and we also know that $t_n = t_1 + (n-1)d$
so if we put this formula for t_n into the sums formula we get
$s_n = (n/2)[t_1 + [t_1+(n-1)d]$
which can be tidied up and simplified to
$\mathbf{s_n = (n/2)(2t_1 + (n-1)d)}$.
here is an example using this version of the formula:
Find s_{20} for the AP {1, 3, 5, ...}

$n=20$, $t_1 = 1$, $d=2$ and so $n-1 = 19$:
$s_{20} = (20/2)(2x1 + 19x2)$
$s_{20} = 10(2 + 38)$
$s_{20} = 10(40)$
$s_{20} = 400$.

Some polygonal numbers



Exercise 7

Use the second version of the s_n formula to find the sum asked for
1) {1, 9, 17, ...} find s_{16}
2) {40, 39½, 39, ...} find s_{21}.

In our formula for s_n there are five items: s_n, n, t_1, d and t_n. If we know four of these then we can be asked to find the fifth.
Most problems can be reduced to two stages:
1) using $t_n = t_1 + (n-1)d$ and
2) using $s_n = \frac{n}{2}(t_1 + t_n)$
(in some cases using the second version of s_n reduces the work to one stage).

We could start with stage 2 and being given s_n have to work backwards to get one of the other items. In one case this leads us into work which requires more advanced algebra techniques (and so it's relegated to the Appendix).

The next exercise follows with a mixed bag of problems

Exercise 8

1) If $t_1 = 21$, $t_n = 193$ and d=2, find s_n.
2) A man travels 40 miles in an hour and then travels 2 miles less in each hour that follows. How far will he travel in all in six hours?
3) A weight-lifter starts off with 100lb to lift and increases this by 10lb each time he lifts the bar. What weight will he have lifted in all after five lifts?
4) Find the sum of the whole numbers bertween 100 and 500 which are divisible by 9.
5) The sum of the terms of {3 + ...+ 59} is 465. What is the average of these numbers and how many are there?
6) A man earns £1,000 in his first year at work and gets a rise each year after that of £100. How much will he earn in all during 12 years?

Exercise 8 continued

7) Find the sum of the integers between 65 and 200 which are not multiples of 6.

8) The sum of 20 terms of an AP is 1280 and the last term is 121. What is the average of the 20 terms?

9) A boy earns £12·50 in his first week and gets a rise of 50p a week for each week after the first. What does he earn in all in 12 weeks of work?

10) A man works for 13 years and his salary for the 13th year is £3,200. If he received a regular rise of £100 every year how much did he earn in all in those 13 years?

11) The trapezium shaped surface of a roof has 30 tiles in the top row. Each row below has one tile more than the row immediately above. If there are 28 rows how many tiles are there in all?

12) A clerk's commencing salary is £2,000 a year andhe is offered a choice between a rise of £50 a year or £220 every four years. Which scheme would you recommend him to choose?

13) Find the first five numbers in an AP if $s_5 = 315$ and $t_5 - t_1 = 28$.

14) A man earns £2,000 in his first year and £3,050 in his eighth year. What does he earn in all in those eight years?

15) From a piece of wire 60″ long 25 pieces are cut off, each 0·1″ longer than the previous one. If the wire completely used up what was the length of the first piece cut off?

16) A man earns £4,000 in his first year and during 10 years earns £49,000 in all with a regular fixed annual increase. What was his salary for the 10th year?

17) A shop sells various sizes of tin kettles and the prices of the successive sizes rise by equal amounts. The smallest costs 10p and the largest 40p. If it costs £2·50 to buy a full set of one of each size, how many sizes are there?

18) Find the sum of all the integers less than 600 which cannot be divided without remainder by 2 or by 3.

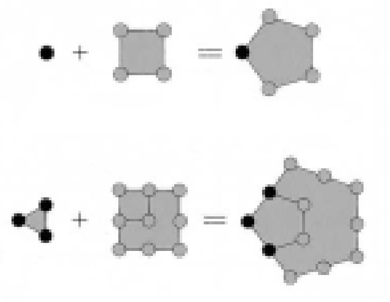
19) Calculate: $(401 + 403 + \ldots + 499) \div (1 + 3 + \ldots + 99)$

20) If $n=10$ and $s_5=7$ and you are also told that the sum of the last 5 terms of an AP is 12, find t_1 and d.

21) In an AP s_4=28 and the greatest term is of value 22. Write out the terms of the series.
22) In an AP n = 21 and the sum of the last three terms is 237; the sum of the three middle terms is 129. What is the AP?
23) A man earns £6,800 in his 10th year at work. In ten years, receiving a regular amount as his yearly rise, he earns £59,000 in all. What was his starting salary and how much was the annual rise?
24) A problem taken from an 1847 text-book states: one hundred stones being placed on the ground at the distance of a yard from one another, how far will a person travel, who shall bring them, one by one, to a basket placed at the distance of one yard from the first stone?
25) A man earns £1,200 in his first year and £2,250 in his last year when he worked for a certain firm. If in that time he earned £13,800 in all for how many years did he work for that firm?
In these problems the second formula for s_n needs less work than the first:
26) s_{10} = 100, n = 10, d = 2. Find t_1 and t_{10}.
27) A man earned £52,000 in all in 16 years. His annual rise was £100. How much did he earn in his first year and how much in the last?

SOME FURTHER IDEAS.

In exercise 3 we showed that pairs of triangle numbers can be added to give the square numbers.
What do you think these diagrams can tell us about polygonal numbers?



The second line shows the pattern better than the first. It can be read as:

Triangle number 2 + square number 3 = pentagon number 3
and the first line is:
Triangle number 1 + square number 2 = pentagon number 2.

Adding a triangle number to a square number produces a pentagon number. Would this be true for other triangle and square numbers?

Try it for the second triangle + third square and for the third triangle and fourth square.

This suggests another question: is there a similar pattern for other polygonal numbers - for example, can we say that the first square number plus the second pentagon number gives us the second hexagon number?

ANOTHER THOUGHT

Here are the cubes (the third powers) of the first five counting numbers:

1, 8, 27, 64, 125.
Try adding these as we did with the terms of the AP.

Number	cube	sum	total
1	1	1	1
2	8	1+8	9
3	27	1+8+27	36
4	64	...	...
5	125	...	...

The last column has three square numbers - 1^2, 3^2, 6^2, the squares of 1, 3 and 6 and this is a set we should recognise {1, 3, 6, ...}; the set of triangle numbers. Do your answers for lines 4 and 5 fit this pattern?

Can we find other patterns like this by adding other powers of the counting numbers? or adding powers of the terms of other sets of numbers?

Can we, for example, find a formula for the sum of the squares?

And connected with this idea is the question of finding general formulæ for the polygonal numbers.

Let's take just one example; the triangle numbers.

These come from adding the terms in the AP of counting numbers, in which $t_1 = 1$, $d = 1$and the last term $t_n = t_1 + (n-1)d$ or $1 + (n-1)$, i.e. just n..

Each triangle number is the sum of a given number of terms of the AP; the formula for the sum of n terms is $s_n = n/2(t_1 + t_n)$ and putting the values of t_1, t_n and d into this we have
$s_n = {}^n\!/\!_2(1 + n)$ or $s_n = \frac{1}{2}n(n+1)$.

Check a value. The fourth triangle number is 10. Put n=4 into this new formula:

$s_4 = \frac{1}{2}(4)(4+1)$
$= 2(5)$
$=10$.

Now we have a formula for the triangle numbers.

This idea can be extended and by taking suitable values for t_1 and d we can find formulæ for other polygonal numbers. try using the idea to find a formula for the square numbers and for the pentagonal numbers.

To finish off here are some problems that need a bit of thought.

Exercise 9

1) In this series **1**, 5, **9**, 13, 17, 21, **25**, 29, 33, 37, 41, 45, **49** ... the numbers in bold type are squares. What is the next square in this series? (Should we assume there will be one?) Do these squares form an easily-recognised pattern?
2) Find the number of terms (n) if the value of s_n for the AP {2, 5, 8,...} is the same as the value of s_n for the AP {47, 45, 43, ...}

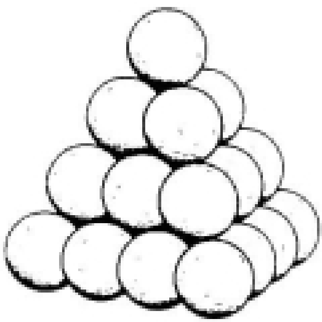
3) In a certain AP the eighth term is twice the fourth term. Show that the ninth term is three times the third term.
4) The houses in a row are numbered consecutively from 1 to 49. One house is empty. The sum of the all house numbers before the empty house is equal to the sum of the house numbers of all those that follow it. What is the number of the empty house?
5) In the next row there are 8 houses, also numbered consecutively. If the same conditions apply, which house is empty?
You might also like to consider what would happen if the houses were numbered with odd numbers, as usually happens.
6) The problem set in questions 4 and 5 can be solved for a row of 8 houses and for a row of 49 houses. Can you find the next number of houses this problem could be set for?

We don’t pretend tha above will help you sleep at nights; but they should give you something to think about.

EXCURSION INTO THREE DIMENSIONS

THE PYRAMID NUMBERS.

We began our investigations into number pattern by looking at



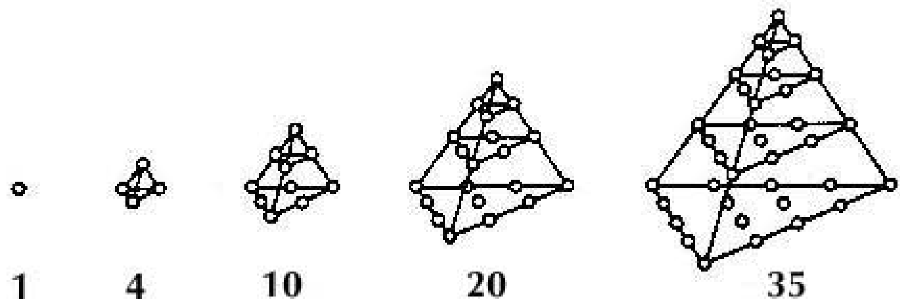
This represents a pile of cannonballs. Problems based on such piles were important to the army in the past and textbooks of the period often have problems concerning pyramids made of cannonballs.

APs which can be drawn as dot graphs - “one dimensional” patterns, and then continued by looking at the sums of the terms in the APs and the polygonal patterns produced by them - a series of two-dimensional shapes.

Now we’ll extend these by having a brief look at patterns in higher dimensions.

Obviously it would be very convenient for a quartermaster if he could calculate the number of cannonballs in such a pyramid. If you have a formula for a pyramid built in this way then you can quickly work out how many cannonballs there are in the pile. It also shows that an abstarct idea like a pyramid number can have a practical application.

Pyramid numbers can also have triangular bases - the sums of the terms in the set of triangle numbers {1, 3, 6, 10, ...}. (Note that uniform symmetrical pyramids are produced only when the pyramids have triangular or square bases.)



If you've followed the development of these ideas so far you may realise that we could go on producing new sets of numbers by taking the sums of the terms in the set of pyramid numbers (3 dimensional) to produce 4 dimensional - and so on, until we get fed up.

Breathe again; we can't draw diagrams for figures of four dimensions and higher, so we'll content ourselves with mentioning the possibility of extending our ideas. Those of you with nervous dispositions should feel suitably relieved.

And finally,

The type of problem we have not considered in the main text is that where we have to find not only the first or last term of an AP but also the number of terms. This is looked at in the Appendix.

Appendix.

Finding n.

Suppose we know that $s_n = 400$, that $d = 2$ and $t_1 = 1$.

We can find $t_n = t_1 + (n-1)d = 1 + (n-1)2 = 1 + 2n-2 = 2n-1$.

$s_n = (n/2)(t_1+t_n) = (n/2)(1+2n-1) = n/2(2n) = n^2$.

$s_n = n^2$.

So $s_n = n^2$ and this is 400; $\mathbf{n^2 = 400}$.

By guesswork, or trial and error, or just by looking back to what we have done earlier you should know that $n = 20$ since $20^2 = 400$.

It might occur to some of you that $(-20)^2$ is also 400. Which makes $n = 20$ and $n = -20$. We haven't said what a negative value for n would mean, but our result implies that there are two answers for 400, i.e. s_{20} and also $s_{(-20)}$. Think about it.

Here's another example. Suppose we know that $s_n = -3$, that $t_1 = -2$ and $d = 1$. What is the value of n?

Using the other s_n formula for a change

$s_n = n/2(2t_1 + (n-1)d)$

$-3 = n/2(-4 + n-1)$

$-3 = n/2(n-5)$

and if I double both sides of this equation to get rid of the ½

$-6 = n^2 - 5n$.

How can we solve this expression? At this stage you'll have to try guessing the answer if you haven't done enough Algebra to find it by algebraic methods.

There are in fact two values of n that will satisfy this equation; try 2 and 3 for n in the expression n^2-5n:

if $n = 2$ then we have $2^2 - 5(2) = 4 - 10 = -6$

if $n = 3$ then we have $3^2 - 5(3) = 9 - 15 = -6$.

The first three terms of the AP are -2, -1 and 0 and the values of s_2 and s_3 are both -3.

This type of problem you can solve by trial and error. The proper method is by using techniques developed for solving Quadratic Equations - a topic beyond the scope of this workbook.