BASE PATTERNS

A study of the patterns of reciprocals expressed in different bases of numeration.

The word fractional is introduced here to describe the representation of a fraction in positional notation; a fraction expressed in base ten is a *base ten fractional* - or **decimal**; in base twelve a *base twelve fractional* - or **dozenal**.
The symbol ↊ is used for ten in base twelve, and the symbol ↋ for eleven.

PART ONE: Descriptive.

1 Types of fractional
Given a fraction 1/a expressed as a fraction in any base of numeration:

1.1 if a contains only the prime factor(s) of the base in use, the fractional will terminate; for example 1/2 = 0·5 base ten, 1/25 = 0·04; 1/3 = 0·4 base twelve.

1.2 if a contains primes which are not factors of the base, and none which are, the fractional will consist of one or more figures repeated in a pattern; this pattern of repeating figures is termed the period of 1/a; for example 1/3 = 0·33333... in base ten, 1/5 = 0·2497 2497 ... in base twelve.

1.3 if a contains primes which are factors of the base as well as some which are not the fractional consists of a repeating pattern preceded by figures which do not recur; such a fractional is termed mixed; for example 1/22 base ten = 0·045 45 45...

In this study we are concerned primarily with the fractionals described in (2) above.

2 Pure recurring fractional patterns
The number of digits in a recurring period varies according to the base used. A prime p has a maximum of (p-1) digits in its recurring period (see 6); where the period is not the maximum (p-1), it is a factor of (p-1) [see 6(2)]

Table 1 shows the number of digits in the periods of the primes 3 to 19 in the bases 2 to 20· The symbol * in the table means the period is the maximum of (p-1) digits; the symbol - means the prime is a factor of the base. There is a sequence for each prime when listed against consecutive bases:

B	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	*	-	1	*	-	1	*	-	1	*	-	1	*	-	1	*	-	1	*
5	*	*	2	-	1	*	*	2	-	1	*	*	2	-	1	*	*	2	-
7	3	*	3	*	2	-	1	3	*	3	*	2	-	1	3	*	3	*	2
11	*	5	5	5	*	*	*	5	2	-	1	*	5	5	5	*	*	*	5
13	*	3	6	4	*	*	4	3	6	*	2	-	1	*	3	6	4	*	*
17	8	*	4	*	*	*	8	8	*	*	*	4	*	8	2	-	1	8	*
19	*	*	9	9	9	3	6	9	*	3	6	*	*	*	9	9	2	-	1

(Note last entry of the pattern is 1)

We will look at these fractionals as follows:
2.1 fractionals for single primes (sections 3 to 5)
2.2 fractionals for powers of a single prime (see 8)
2.3 fractionals for products of primes - i.e. composite numbers.

If the period is a maximum then the digits in the fractional circulate, i.e. multiples of the simple fractional repeat the sequence of digits of the simple fractional; if the period is less than the maximum there will be more than one circulating pattern.

Examples (the form 1/a decimal throughout)

1.1	1/2 =	0·5 base ten; 1/4 - 0·4 base sixteen; 1/9 = 0·14 base twelve;
	1/12 =	0·19 base eighteen
1.2	1/7 =	0·142857 base ten; the six-figure group repeats
	1/21 =	0·047619 base ten; the six-figure group repeats
	1/11 =	0·1111 base twelve; the figure 1 repeats
	1/55 =	0·0275 base twelve; the four-figure group repeats
1.3	1/30 =	0·0 333 base ten; the 3 repeats
	1/105 =	0·0 095238 base ten; the six-figure group repeats
	1/12 =	0·08 3333 base ten; the 3 repeats
	1/84 =	0·01 190476 base ten; the six-figure group repeats.

and examples of those in groups 2 and 3:

2.1.1 Sevenths, bases ten and five (142857 stands for 0·142857 142857 ...)

1/7	142857	032412...
2/7	285714	120324
3/7	428571	203241
4/7	571428	241203
5/7	714285	324120
6/7	857142	412032

2(1)(2) 1/13 base ten, which has two 6-figure patterns:

13ths	pattern a	pattern b
1	076923	
2		153846
3	230769	
4	307692	
5		384615
6		461538
7		538461
8		615384
9	692307	
10	769230	
11		846153
12		923076

1/7 base 9, with two three-figure periods:

sevenths	pattern a	pattern b
1	125	
2	251	
3		376
4	512	
5		637
6		763

2.2.1 1/25 (= $1/5^2$), base twelve; twenty figures in period,
circulating, with patterns for the fifths (4 multiples):
25ths
1 05915 343↊0 ↋62↊6 8781↋ recurring
2 0↋62↊ 68781 ↋0591 5343↊
3 15343 ↊0↋62 ↊6878 1↋059
4 1↋059 15343 ↊0↋62 ↊6878
5 2497 2497 ... (= 1/5)
There are four multiples of 1/5 and twenty multiples of twenty-fifths (not counting 25/25 = 1). Twenty figures is the maximum period for 1/25 in any base (see 8).

2.2.2 In base 9 1/25 has three patterns;
there are the multiples with basic pattern 0321385675,
those with 0642782461 and the fifths with pattern 17 (i.e. 1/5 = 0·17 17 17...)
2.3.1 In base eight 1/15; there are two patterns: 0421 and 3567;
there are also the patterns for the thirds: 25 and for the fifths: 1463.
2.3.2 In base four: 1/21: there are five patterns, of three figures; for the 21sts 003, 012, 033, 123 and 132; there are the patterns for the thirds:111, and sevenths: 021.

Mixed recurring fractionals: if p is a number prime to the factors of the base f(see p3), then the form yields a mixed circulator with either m or n non-recurring figures before the recurring group for p, according as m or n is the greater.
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3. Reversing properties

1) Using the two-way (or Reverse) notation, some reciprocal periods split into two parts, one of which is the negative (or reverse) of the other; fractionals which yield such forms are those which are the reciprocals of
(a) a prime,
(b) a product of primes none of which is capable separately of giving the period that the product gives, and
(c) mixed versions of the above.

The period of the product of two or more primes is the LCM of the periods of the primes in the base used (see 9). If the first n figures of a prime fractional are known, the second n, where the fractional has 2n figures altogether, can be obtained from the first by subtracting the first n, in order, from (r-1) where r is the base used.

Example 1/7th base ten: 0·142857...
142 and 857: 1+8 = 4+5 = 2+7 = 10-1 = 9.
1/7th base twelve: 0·186Ꮖ35...
186 + Ꮖ35: 1+Ꮖ = 8+3 = 6+5 = 10-1 = Ɛ.

In two-way notation 0·143 857... becomes $0{\cdot}14\bar{3}\ 14\bar{3}$ ...

The first dozen numbers in base ten reverse notation are:
1 2 3 4 5 $1\bar{4}$ $1\bar{3}$ $1\bar{2}$ $1\bar{1}$ 10 11 12 ... i.e. we count backwards and forwards from the multiples of ten.

The reverse notation can obviously be used in any base; it is at its most efficient when used in an odd number base, since in an even number base the half-base (e.g. 5 in base ten) has two forms
(5 and $1\bar{5}$).

In base 7, for example, the sequence of numbers is:
1 2 3 $1\bar{3}$ $1\bar{2}$ $1\bar{1}$ 10 11 12 ... there is no half-base, and the forms 3 and $1\bar{3}$ represent two different numbers (3 and 4).

The even-number bases sometimes disguise the reversing pattern of a fractional due to the double representation of the half-base; patterns may have more than one form when written in reverse notation.

References:
The Reverse Notation, by J.Halcro Johnston (Blackie) in which the notation is described and applied to the dozenal system.
Biomathematics vol 2, by C.Smith (Ch. Griffin & Co); (referred to as Colston Notation)
An Introduction to Number Scales & Computers, F.J.Budden, pp112-3,(Longmans).

Note on the reverse notation:
the notation can be regarded also as a representation of operations performed on a hand-driven calculating machine (most of which will probably be in museums nowadays...).
In base ten. for example, we do not need to turn the operating handle nine times forwards to multiply a number by 9, but instead multiply by (10 - 1), i.e. turn once forwards in the tens position and once backwards in the units position; this corresponds to the reverse notation for 9 in base ten: 11.
As a further example, suppose we are multiplying by 17789 - which would require 32 forward movements if we could not move the carriage and turn the handle backwards. The reverse notation for 17789 in base ten is $2\bar{2}\bar{2}\bar{1}\bar{1}$, or 8 movements in all, two forwards and six backwards.

3(3)
The fractionals discussed under section 2 have reversing properties; they fall into the following groups:
1) fractionals of full period, which reverse within themselves;
2) fractionals with less than the full period, which can be further subdivided into:
a) those in which the pattern for a multiple reverses within
itself
b) those in which the pattern does not reverse within itself, but
has a reverse form in some other multiple.

Examples
3(3)1 1/7 base 5 = 0·032412... = 0·112 112 (112 and reverse)
3.3.2
a 1/13 base ten, pattern a: 076923 = $1\bar{2}\bar{3}\ \bar{1}23$ ...
b: 153846 $154\ \bar{1}\bar{5}\bar{4}$ ...
b 1/41 base ten
The maximum number of figures for 1/41 in any base is 40·
In base ten we have five figures (one-eighth of the full period) and find eight distinct patterns, each of which contains five multiples.

Pattern	for multiples		reverse form
02439	1 10 16 18 37	a	02439 $0244\bar{1}$
04878	2 20 32 33 36	b	04878 $15\bar{1}\bar{2}\bar{2}$ b
07317	3 7 13 29 30	c	07317 $1\bar{3}32\bar{3}$ c
09756	4 23 25 31 40	a	09756 $10\bar{2}\bar{4}\bar{4}$ a
12195	5 8 9 21 39	b	12195 $122\bar{1}\bar{5}$ b
14634	6 14 17 19 26	d	14634 $2\bar{5}\bar{4}34$ d
26829	11 12 28 34 38	c	26829 $3\bar{3}\bar{2}3\bar{1}$ c
36585	15 22 24 27 35	d	36585 $4\bar{3}\bar{4}\bar{2}5$ d

corresponding reverse forms are labelled a, b, c, d - for example 02439 and 09756 have connected reverse forms; a fraction greater than half is expressed in reverse notation as 1 minus a fraction less than half; 40/41, for example, is 1 -(1/41), the pattern for 1/41 linking the two. 1/41 = 0·02439... = $0{\cdot}024\bar{4}\bar{1}$...
40/41= 1 - $0{\cdot}024\bar{4}\bar{1}$... = $1{\cdot}0\bar{2}\bar{4}41$...
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4) Patterns in fractionals

1) The digits of the fractionals show a consistent pattern for a given form of base. Example: 1/5 in bases 2, 7 and 12. The numbers 2, 7 and 12 are of form 5n+2; [we could also describe them as being equivalent to 2 in mod 5].

n	base 5n+2	one-fifth
0	2	0·0011 0011 ...
1	7	0·1254 1254 ...
2	12	0·2497 2497 ...

Calling the four digits of the period a, b, c and d, we can see that for n=n: a=n, b=2n, c=4n+1 and d=3n+1.
We can express this, for convenience, as 0·(n)(2n)(4n+1)(3n+1)...
[the brackets being used to separate the terms] or alternatively as
0·(1,0)(2,0)(4,1)(3,1)...

2)
Similar patterns can be established for other primes and other forms of bases (shown in table 2). We can find them, as above, by observation, or by division:

1/5 in base 5n+2:
(5n+2)/ 5 = n, remainder 2
2(5n+2) = 10n+4; (10n+4)/ 5 = 2n, remainder 4
4(5n+2) = 20n+8; (20n+8)/5 = 4n+1, remainder 3
3(5n+2) = 15n+6; (15n+6)/5 = 3n+1, remainder 1 ...
Using reverse notation this structure becomes
0·(n)(2n+1)(-n)(-2n-1)... which can be abbreviated as
0·(n)(2n+1)&r, where &r means "and reverse".
Alternatively 0·(1,0)(2,1)$(\bar{1},0)(\bar{2},\bar{1})$... 0·(1,0)(2,1)&r.

Table 2 Examples
1/p has the fractional form listed in the table for a base of form pn+k. The forms given are those for reverse notation.

pn+k	fractional
5n+2	0·(1,0)(2,1)&r
5n-2	0·(1,0)(2,1)&r
7n+3	0·(1,0)(3,1)(2,1)&r
7n-3	0·(1,0)(2,1)(3,1)&r
11n+2	0·(1,0)(2,0)(4,1)(3,1)(5,1)&r
11n-5	0·(1,0)(5,2)(3,1)(4,2)(2,1)&r
11n-4	0·(1,0)(4,1)(5,2)(2,1)(3,1)&r
11n-3	0·(1,0)(3,1)(2,1)(5,1)(4,1)&r

and, for comparison, forms that do not reverse:
(for k = +-1 see later, section 5.4...)

7n-3	0·(1,0)(3,1)(2,1) repeating
7n+2	0·(1,0)(2,1)(3,1) repeating
11n-2	0·(1,0)(2,0)(4,1)(3,1)(5,1) repeating
11n+5	0·(1,0)(4,1)(5,2)(2,1)(3,1) repeating
11n+3	0·(1,00(3,1)(2,1)(5,1)(4,1) repeating

5) In general

1) The multiples of circulating fractionals will contain exactly the same pattern only if the fractional period contains exactly the same number of digits possible.

2) If a fractional be divided by a prime factor of the base used, the resulting fractional will also circulate and will share the reversing properties of the original fractional. (section 1.3)

3) If the fractional be divided by any other prime the resulting period will be of n digits, where n is the LCM of the periods of the fractional and the prime in the given base; the resulting expression may or may not reverse and may or may not circulate.

4) A fractional which circulates and reverses in one base will do so in another base of the same form pn+k, where p is the prime concerned, but not necessarily in another base. (Values of k are discussed in section 7).

5) If a fractional period is less than the maximum with an even number of digits, it may reverse; if it has an odd number of digits, it will not reverse.

Examples for 5(4)
1/7 expressed as a fractional in bases two to fourteen

base	repeating pattern	digits	form of base
2	001	3	7n+2
3	010212	6	7n+3
4	021	3	7n-3
5	032412	6	7n-2
6	05	2	7n-1
7	- (exact)		7n
8	1	1	7n+1
9	125	3	7n+2
10	142857	6	7n+3
11	163	3	7n-3
12	186↊35	6	7n-2
13	1↋	2	7n-1
14	- (exact)		7n

Notes:
The form pn-1 for prime p always yields two figures 0·(n)&r in reverse notation. The fractionals for 1/(a+1) in base s is 0·0(a-1); this is the form when we take n=1 in the base form pn-1.
The form pn+1 yields 1/p = 0·(n) with n repeating. In the list of forms for 1/7 it will be noted that base forms 7n+3 and 7n-2 yield full period; (see also 7(2); the values +3 and -2 are primitive roots mod 7).

Further examples
showing patterns of multiples for 1/7 in selected bases:

	2	3	4	6	8	9	11	12
1	001	010212	021	05	1	125	163	186↊35
2	010	021201	102	14	2	251	316	35186↊

3	011	102120	123	23	3	376	479	5186↊3
4	100	120102	210	32	4	512	631	6↊3518
5	101	201021	231	41	5	637	794	86↊351
6	110	212010	312	50	6	763	947	↊35186
	7n+2	7n+3	7n-3	7n-1	7n+1	7n+2	7n-3	7n-2

Patterns

	2	3	4	6	8	9	11	12
1	a		a	a		a	a	
2	a		a	b		a	a	
3	b		b	c		b	b	
4	a		a	c		a	a	
5	b		b	b		b	b	
6	b		b	a		b	b	

(period-length)x(number of periods) = (7 - 1) = 6

2. Fractionals as geometric progressions.
Any fractional can be regarded as a geometric progression and its value as a fraction can be found by taking the sum to infinity of the GP
1/7 base ten
first term of GP = a = 0·142857
common ratio, r = 0·000001
sum = a/(1-r) = 0·142857/(1-0·000001) = 142857/999999 = 1/7.

3. Fractionals as series
In the above example we could regard the fractional instead as

142857(0·000001 000001 ...) i.e. the series

142857 x $\sum 1/10^{6s}$ and the series evaluated as 142857x1/999999

Using reverse notation the fractional becomes $0{\cdot}143\bar{1}\bar{4}\bar{3}$... which can be expressed as $143(0{\cdot}001\ 00\bar{1}$ ...) or
143∑(-1)^(s-1)/10^3s or 143(1/1001).

4. Reciprocal Series Formula
The remarks in section 3 above can be applied to any fractional in any base. From the Binomial Theorem we can derive the formula

$$\frac{1}{10^t - m} = \sum_{s=1}^{s \to \infty} \frac{m^{s-1}}{10^{st}}$$

which we can call the Reciprocal Series Formula (RSF), a series which lends itself to calculation in any base. Note that the symbol “10” in this formula represents any base we choose.

Example 1/5 base 7
The symbol “10” represents 7; 1/5 = 1/(10-2); m=2. Powers of 2 in base 7 are 2, 4, 11, 22 ...
from the RSF 1/5 = 0·1 + 0·02 + 0·004 + 0·0011 + ...
which, eventually, leads to 0·1254 1254 ...

1/7 base ten; m=3

Term n	sum to n terms
1	0·1
2	0·13
3	0·139
4	0·1417
5	0·14251
...	
12	0·142857066937 and so on.

The formula excels where the power of ten is high and the quantity m small.
1/ƐƐ7 base twelve = 1/(1000-5) = 0·001 005 021... and the size of the power gives us a large number of fractional places quickly.

The value m can also be negative and which allows us to calculate $1/(10^t+m)$ as $1/[10^t-(-m)]$, i.e. using (-m) in the formula instead of m.

1/13 base ten 13 = 10+3, m = -3.
0·1 - 0·03 + 0·009 -0·0027,
$0{\cdot}1 - 0{\cdot}03 + 0{\cdot}01\bar{1} - 0{\cdot}003\bar{3}$, or $0{\cdot}1 - 0{\cdot}03 + 0{\cdot}01\bar{1} + 0{\cdot}00\bar{3}3$ ...
Many other reciprocals can be adapted to a form suitable for the RSF.
For example 1/49, base ten: 1/49 = 2/98, and we can use the series for 1/98.
Alternatively, $1/49 = (1/7)^2 = 1/(10-3)^2$
or $1/10^2.(1-3/10)^2 =$

i.e. $1.1/10^2 + 2.3/10^3 + 3.9/10^4 + \ldots$
0·01 + 0·006 + 0·0027 + 0·00108 +...
0·020408...

5.6 reciprocals as series

1 Converting fractional to fraction

Multiply the fractional by a power of the base, and subtract to remove the recurring pattern.
For example, 1/9 base 10, is 0·11111....
f = 0·11111...
10f=1.11111...
subtracting, 9f = 1, or f = 1/9
If there are n digits in the period, multiply by 10^n:
Example, base ten:
f = 0·09 09 09 ...
100f= 9.090909 ...
subtract: 99f = 9, so f = 9/99 = 1/11.
in reverse notation:
f=0·1$\bar{1}$ 1$\bar{1}$ 1$\bar{1}$ ...
10f=1.$\bar{1}$1 $\bar{1}$1 $\bar{1}$1 ...
ADD: 11f = 1, so f = 1/11. (Add, to cancel out the positive and negative digits).
Example, base twelve:
f = 0·186↊35 ...
10^6f=186↊35 186↊35....
(10^6-1)f= 186↊35, so f = 186↊35/ƐƐƐƐƐƐ = 1/7.
and
f = 0·2$\bar{3}\bar{5}$ $\bar{2}$35 ...
10^3f = 2$\bar{3}\bar{5}$.$\bar{2}$35 2$\bar{3}\bar{5}$...
add: (10^3+1)f = 235; so f = 235/1001 = 1/7. (2$\bar{3}\bar{5}$ = 187)