Find the next term ...

by Shaun Ferguson

"Find the next number in the sequence ..." is a favourite problem set by newspaper "Puzzle Corners" and numerically-minded Quiz Shows. This activity ranges from nice simple patterns such as *2, 4, 6, 8...* to quite complicated structures. Implied in the request to find the next number is the idea that there *is* a next number and that it is *unique*. Can we assume so much? Is there a unique answer - or are there sequences that admit of more than one solution, that are in fact 'ambiguous' ? If so, now many - or how few - more terms of the sequence do we need to make it unambiguous?

Given the sequence 2, 4, 6, 8 ... most people would assume that the next number is ten - it is the most obvious answer. I have written "ten" on purpose - after all, we could use a base other than ten, in which the number ten itself would appear in a different guise - in base nine, for example, ten is written "11" (one nine and one unit) and in base twelve it would need a single digit because in base twelve, twelve itself is written "10". We'll only use one base here.

All the number work that follows is in base twelve - if you've met the logo above before you'll know that it's one I use on work for the DSGB (Dozenal Society of Great Britain).

In base twelve "10" means one dozen and no units, so we need two new symbols - one for "ten" and one for "eleven". I use an inverted 2 for ten and an inverted 3 for eleven, which gives us the sequence:

0 1 2 3 4 5 6 7 8 9 ↊ ↋ 10 11 12 13 14 15 16 17 18 19 1↊ 1↋ 20 ...

To return to our number patterns. How few terms do we need to fix the pattern? Three? As in the sequence 1,2,3,.. or 2,4,6,.. But even the innocent-looking 1,2,3... is wide open - has the 3 been produced by adding a unit? Or is it, as in the Fibonacci sequence, the sum of the first two terms? The pattern could go on 1,2,3,4,5,6,... or (Fibonacci) 1,2,3,5,8,11,19... (No, 11 is not a mistake; it's still what we usually call thirteen - but here's it is "a dozen and one", 11). So three terms, for the sequence 1,2,3.. are not enough - maybe four would fix the pattern.

What sparked this article off was a remark that the sequence 2,3,5.. could be taken as the start of the set of prime numbers (2 being the only even one). Admittedly they could be the primes - in which case
a) 2 3 5 7 ↋ 11 15 17 is the sequence intended; or
since 2 x 3 -1 =5 it might be
b) 2 3 5 12 59 (12 = 3x5 -1; 59 = 5 x 12 -1 etc); or
with the nth term being $2^{n-1} + 1$ it might be
c) 2 3 5 9 15; or
there could be a pattern of increasing addends:
d) 2 3 5 8 10 15 (2 + **1** =3; 3 + **2** = 5; 5+ **3** = 8; 8 + **4** = 10 etc)
and if you note that the first four terms of pattern (d) are the same as those of the Fibonacci sequence
e) 2 3 5 8 11 19 ...
maybe we need at least 5 terms...

Well? Can you write down a sequence of numbers such that there is only one (unambiguous) value for the term after the last one written down? Just the numbers, not a rule stated in algebraic or set-theoretic terms. No, you can't even have 2, 4, 6, 8... attractive though the thought may be. The next number *might* be **ten,** but it could equally well be Ɛ or 10... Another valid sequence is
(a) 2 4 6 8 10 1Ƨ.. and another (b) 2 4 6 8 Ɛ 15...
Can you construct the rule I've used here?
And here are two more - what's the next term for these?
c) 1, 2, 4, 8, 14, 26, 44 ...
d) 1, 2, 4, 8, 14, 28, 5Ƨ...

Hints follow on the next page.

The 2, 4, 6, 8, ... patterns are created by:

a) $f(n) = 2n + (n-4)(n-3)(n-2)(n-1)/10$

and

b) $f(n) = 2n + (n-4)(n-3)(n-2)(n-1)/20$

and similar ideas created the other two patterns (c) and (d). [I think (c) is connected with the number of segments in a circle cut by n intersecting lines.]

And to finish, in (d), $5 \times 12 - 1 = 59$ is true in any base b, where $b > 9$.

(Adapted from my article in the Dozenal Bulletin of the Dozenal Society of America)