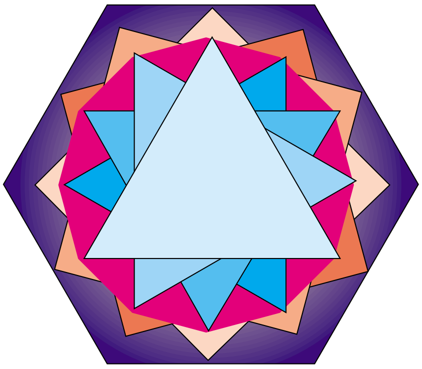

Divisibility

Rules and recipes
for testing divisibility
in different number bases

In our most familar number base, base ten, we can find out if a number can be divided exactly by 9 by simply adding up the digits of the number; if the total we get can be divided exactly by 9 then so can the number. This test for 9 will not necessarily work in a different number base, but the form of this test ("add up the digits") can be used for some other number. For example, given a number written in base seven, if the digit total can be divided by 6, then the number can also be divided by 6.

The expression "the sum of the digits of the number" is long winded; there is no single word for it in English, but in German there is the word "**Quersumme**"; so we will use **Q(*x*)** to mean "the sum of the digits in the number *x*". Q(123) = 1+2+3 =6. Note that this is not the same as what is called the *"digital root"* of a number.

In any base, r, this "add digit" rule applies to the number (r-1). In base ten, this number is 9, and factors of 9, such as 3, share the same rule. If in base ten Q(*x*) has the factor 3, then *x* is a multiple of 3. In base seven the rule applies to 6 (=7-1) and factors of 6, such as 2 and 3; this in turn tells us that a number *x* is even in base seven if Q(*x*) is even. In any odd number base if Q(*x*) is even then so is *x*.

There is also the "alternate digit" rule - which tests divisibility by (r+1) in base r. In base ten (r+1) is eleven; we can tell if a number is divisible by eleven if the difference between the total of the digits in the even places and that in the odd places is 0 or a multiple of eleven. Here we will use **A**(x) for this function. For example: 25641 is a multiple of 11 in base ten, because A(25641) = (2+6+1) - (5+4) = 0. Three digit numbers which divide by "11" in any base are usually easy to spot - the middle digit is the sum of the two outer digits - for example, in base ten, 385 = 35 x 11, and 8 = 3+5. If there's a "carry", it won't be so obvious, however: 19 x 11 in base ten = 209, and "59" x "11" in base twelve is "629".

If the number is a factor of the base then all we have to do is look at the last digit in the number: ten divides by 2, so if the number ends in an even digit the whole number is even. In base twelve, by comparison, this rule - looking at the last digit, applies also to 3, 4 and 6 because these are all factors of twelve; I don't need to know the multiplication tables for base twelve unless I want to (they are quite easy). In base seven, on the other hand all we can say is that if the number ends in 0 then it can be divided by 7. Seven is not a factor of ten or twelve, so if I want to check for divisibility by 7 in either of these bases, then I need a different rule.

As an example, look at the rules for 3 in all bases:

We can put the bases into three groups - those of form 3n-1, 3n and 3n+1.

3n-1	**3n**	**3n+1**	
2	3	4	
5	6	7	
8	9	10	
11	12	13	etc

Group "3n": 3 is a factor of the base; look at the last digit of the number.
Group "3n+1": 3 is a factor of 1 less than the base; apply **add digits rule**.
Group "3n-1: 3 is a factor of 1 more than the base; apply **alternate digit rule**.

What properties are common to all bases? I can look at the last digit or digits of a number; I can add up the digits in a number; I can use the difference method. What other ways are there?

These three apparently unconnected ways of testing for divisibility are really three aspects of one basic rule. All rules of divisibility can be linked with **power residues** (see below) and **Fermat's Theorem** (*not* his *Last* one...). They appear to fall into different categories because we select special cases.

If we divide 17 by 3 we find 3 will divide into seventeen five times, with remainder 2; this remainder is the "residue modulus 3" of the number 17. (Modulus is usually abbreviated to *mod*). If we divide 10, 100 or 1000 (the first three powers of ten) by 3 we get a remainder of 1 in each case; these residues of these powers of ten are called "power residues". For example: the power residues of twelve mod 5:

$12^0 =$	1	$\equiv 1 \bmod 5$	or $12^0 \equiv 1 \bmod 5$
$12^1 =$	12	$\equiv 2 \bmod 5$	
$12^2 =$	144	$\equiv 4 \bmod 5$	
$12^3 =$	1728	$\equiv 3 \bmod 5$	
$12^4 =$	20736	$\equiv 1$ beginning the sequence of power residues again.	

(These power residues give us a general method for divisibility in any base).

To summarise:

for a number, N, in any base, r, where we are testing N for divisibility by a number x:

1) x is a factor of r^n	L(x)	Last *n* digits of N
2) x is a factor of r-1	Q(x)	Add the digits
3) x is a factor of r+1	A(x)	Difference of digits

and 4) the general rule: use power residues.

Cases (2) and (3) are part of the general rule, where the power residues are 1 and -1 respectively. Rule 1 is also a special case: if x is a factor of the base then it also a factor of all the powers of the base; so the only digit we need look at is the units digit. If x is not a factor of the base butis a factor of a power of the base then we may need to look at more digits than just the last - for example, in base ten, 8 is not a factor of ten, but it is a factor of 1000 (= 10^3) and we look at the last 3 digits of the number.

Examples of the four rules:

1)

(a) **Look at the last digit**

1213 is divisible by 3 in base twelve because 1213 ends in 3.

(b) **Look at the last (n) digits**

In base ten 4 does not divide ten exactly, but it does divide $(\text{ten})^2$, so we look at the last 2 digits.

132 is divisible by 4 in base ten because 32 is divisible by 4 in base ten.

2) **Add the digits**

246 is divisible by 6 in base 7 because Q(246)= 15 (base seven) and 15 (base seven) is a multiple of 6 [Q(15)=6]. (r=7; r-1=6)

3) **Alternate digits rule**

369 is divisible by 6 in base eleven because A(369) = 6 and this is obviously divisible by 6. (r=eleven; r+1 = twelve, a multiple of 6).

4) **Power residues**

Five is not a factor of twelve, so there is no obvious quick sight rule to find out if a given number in base twelve is divisible by 5.

We can, however, use the power residues of twelve to test for divisibility by 5.

If we divide twelve by 5 we have the remainder (or residue) 2, and we can write this as

12 = 2 mod 5 (or 12^1 = 2 mod 5)

Then 12^2 = 4 mod 5

12^3 = 3 mod 5

12^4 = 1 mod 5, after which the sequence repeats.

Noting that 12^0 = 1 mod 5, we have the pattern 1, 2, 4, 3.

So, if we have a number, such as 2748, say, in base twelve, we can test for divisibility by 5 by replacing the powers of twelve by these residues.

The number 2748 (base twelve) is made up of:

2 x 12^3 =	**2** x 3 mod 5 =	1 mod 5
7x 12^2 =	**7** x 4 mod 5 =	3 mod 5
4x 12^1 =	**4** x 2 mod 5 =	3 mod 5
8 x 1 =	**3** mod 5 =	3 mod 5
Total	**(1+3+3+3)=**	**0 mod 5.**

The total is 0, so 2748 base twelve is divisible by 5.

These power residues of twelve mod 5 can also be expressed as minimal residues.

3 = -2 mod 5, and 4 = -1 mod 5, so we can also have

2 x 12^3 =	**2** x -2 mod 5 =	1 mod 5	(-4 = 1 mod 5)
7 x 12^2 =	**2** x -1 mod 5 =	-2 mod 5	
4 x 12^1 =	**-1** x 2 mod 5 =	-2 mod 5	
8 x 12^0 =	**-2** x 1 mod 5 =	-2 mod 5	
Total	**(1-2-2-2 = -5)**	**0 mod 5**	

A second leaflet is in preparation which will extend these ideas and introduce other methods of deciding divisibility.